SOME GLOBAL RESULTS FOR
ASYMPTOTICALLY SIMPLE SPACETIMES

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1. Introduction

The aim of this paper is to present some rigidity results for asymptotically simple spacetimes. By a *spacetime* we mean a smooth connected time oriented Lorentzian manifold (M, g) of dimension ≥ 3, having signature $(- + \cdots +)$. We use standard notation for causal theoretic notions, e.g., for $A \subset M$, $I^+(A, M)$, the timelike future of A in M (resp., $I^-(A, M)$, the timelike past of A in M), is the set of all points that can be reached from A by a future directed (resp. past directed) timelike curve in M. For other basic definitions and results in spacetime geometry and causal theory we refer the reader to the standard references [12, 17].

Penrose's treatment of infinity [16, 18] in asymptotically flat spacetimes (and spacetimes with other asymptotic structures, as well) is based on his notion of *asymptotic simplicity*. A 4-dimensional, chronological spacetime (M, g) is asymptotically simple provided there exists a smooth spacetime-with-boundary $(\tilde{M}, \tilde{g})$ such that,

(a) M is the interior of $\tilde{M}$, and hence, $\tilde{M} = M \cup \mathfrak{I}$, where $\mathfrak{I} = \partial\tilde{M}$,
(b) $\tilde{g} = \Omega^2 g$, where Ω is a smooth function on $\tilde{M}$ such that (i) $\Omega > 0$ on M and (ii) $\Omega = 0$ and $d\Omega \neq 0$ along $\mathfrak{I}$, and
(c) every inextendible null geodesic in M has a past and future end-point on $\mathfrak{I}$.

Condition (c) is a strong global assumption, which implies that M is null geodesically complete, and ensures that $\mathfrak{I}$ includes all of null infinity. But it also rules out spacetimes with singularities, and black holes, etc. To treat such cases, condition (c) must be suitably weakened, cf. [12, 20].

In this paper, we will be primarily interested in asymptotically simple spacetimes which obey the vacuum Einstein equation with cosmological term,

$$(1.1) \qquad\qquad \mathrm{Ric} = \lambda\, g\,.$$

If $\lambda = 0$ (the asymptotically flat case) then $\mathfrak{I}$ is necessarily a smooth null hypersurface, which decomposes into two parts, $\mathfrak{I}^+$, future null infinity, and $\mathfrak{I}^-$, past null infinity. If $\lambda > 0$ (the asymptotically de Sitter case) then we have a similar decomposition, except that $\mathfrak{I}$ is spacelike.

We present here a uniqueness theorem for Minkowski space and for de Sitter space associated with the occurrence of *null lines*. A null line in a spacetime (M, g) is an inextendible null geodesic which is globally achronal (meaning that no two points can be joined by a timelike curve). Arguments involving null lines have arisen in many situations, such as the Hawking-Penrose singularity theorems [12], results on topological censorship [3, 9], the Penrose-Sorkin-Woolgar approach to the positive mass theorem [19, 21] (and related results on gravitational time delay [11]), and most recently results concerning the AdS/CFT Correspondence [10].

Every null geodesic in Minkowski space and de Sitter space is a null line. At the same time, each of these spacetimes, and indeed any spacetime satisfying equation (1.1) (regardless of the sign of λ) obeys the *null energy condition*, $\text{Ric}\,(X, X) = R_{ij}X^iX^j \geq 0$, for all null vectors X. In general, it is difficult for complete null lines to exist in spacetimes which obey the null energy condition. The null energy condition tends to focus congruences of null geodesics, which can lead to the occurence of null conjugate points. A null geodesic containing a pair of conjugate points cannot be achronal. Thus we expect a spacetime which satisfies the null energy condition and which contains a complete null line to be special in some way. The following theorem supports this point of view.

Theorem 1.1. *Suppose M is an asymptotically simple spacetime satisfying the vacuum Einstein equation (1.1) with $\lambda \geq 0$. If M contains a null line then M is isometric to Minkowski space (if $\lambda = 0$) or de Sitter space (if $\lambda > 0$).*

Let us give an interpretaton of Theorem 1 in terms of the initial value problem for the vacuum Einstein equation, in the case $\lambda > 0$. According to the fundamental work of Friedrich [4], the set of asymptotically simple solutions to (1.1), with $\lambda > 0$, is open in the set of all maximal globally hyperbolic solutions with compact spatial sections. Thus, by Theorem 1, in conjunction with the work of Friedrich, a sufficiently small perturbation of the Cauchy data on a fixed Cauchy hypersurface in de Sitter space will in general destroy all the null lines of de Sitter space, i.e., the resulting spacetime that develops from the perturbed Cauchy data will not contain any null lines. While one would expect many of the null lines to be destroyed, it is somewhat surprising that *none* of the null lines persist. One is tempted to draw a similar conclusion in the $\lambda = 0$ case. However, the nonlinear stability of asymptotic simplicity in this case has not been established, and, indeed may not hold, cf., [6, 7] for further discussion. In fact at present, Minkowski space is the only known asymptotically simple solution to (1.1) with $\lambda = 0$ (though evidence is mounting that there are solutions distinct from, but in a suitable sense, close to Minkowski space; by Theorem 1, such solutions would have no null lines). Finally, we note that if M is an asymptotically simple and de Sitter spacetime without any null lines then the sets $\partial J^+(p)$ will be compact for all $p \in M$ sufficiently close to $\mathcal{I}^-$. As all such

sets in de Sitter space are noncompact, this further serves to illustrate the fragile nature of the causal structure of de Sitter space (see also [11, Corollary 1]).

Theorem 1 is a consequence of the *null splitting theorem* obtained in [8]. This latter result establishes the rigidity in general of null geodesically complete spacetimes which obey the null energy condition. We discuss this result in the next section. In section 3 we present the proof of Theorem 1. For results of related interest concerning the rigidity of asymptotically simple spacetimes, see, for example, [1, 5, 13].

2. The Null Splitting Theorem

The statement of the null splitting theorem involves null hypersurfaces. We recall briefly some facts about the geometry of null hypersurfaces; see e.g., [8, 12, 17] for more detailed discussions from slightly varying points of view.

A smooth null hypersurface in a spacetime (M, g) is a smooth co-dimension one submanifold S along which the Lorentz metric g is degenerate. Hence, S admits a smooth future directed null vector field K, which is unique up to a positive scale factor. As is well-known, the integral curves of K, when suitably parameterized, are null geodesics, and are referred to as the null generators of S.

The *null expansion tensor* (or *null second fundamental*) Θ of S measures the variations in the spatial separation of the null generators of S. Let TS/K denote the tangent vectors to S mod K; for each $p \in S$, $T_pS/K = \{\bar{X} : X \in T_pS\}$, where $\bar{X} = \{Y \in T_pS : Y = X \bmod K\}$. TS/K is an $n-2$ dimensional vector bundle over S ($n = \dim M$). For $\bar{X}, \bar{Y} \in T_pS/K$, define $h(\bar{X}, \bar{Y}) = g(X, Y)$; h is a well-defined smooth Riemannian metric on TS/K. Θ is defined as, $\Theta(\bar{X}, \bar{Y}) = g(\nabla_X K, Y)$, where ∇ is the Levi-Civita connection of (M, g); Θ is a well-defined symmetric bilinear form on TS/K, unique up to the scaling of K. By tracing Θ with respect to h, we obtain the null expansion scalar $\theta = h^{ab}\theta_{ab}$, which measures the divergence of the null generators towards the future. Along an affinely parameterized null generator, $s \to \eta(s)$, the null expansion $\theta = \theta(s)$ satisfies (provided K is appropriately scaled) the Raychaudhuri equation for an irrotational null congruence,

$$(2.1) \qquad \frac{d\theta}{ds} = -\mathrm{Ric}\,(\eta', \eta') - \sigma^2 - \frac{1}{n-2}\theta^2\,,$$

where $\sigma^2 = \sigma_{ab}\sigma^{ab}$, and σ_{ab} is the shear tensor, $\sigma_{ab} = \theta_{ab} - \frac{1}{n-2}\theta$.

We say that S is *totally geodesic* if and only if the expansion tensor vanishes, $\theta_{ab} \equiv 0$, or, equivalently, if and only if the expansion scalar and shear vanish, $\theta \equiv 0$, $\sigma_{ab} \equiv 0$. This has the usual geometric meaning: A geodesic in M starting tangent to a totally geodesic null hypersurface S remains in S. Null hyperplanes in Minkowski space are totally geodesic, as is the event horizon in Schwarzschild spacetime.

We now state the null splitting theorem.

Theorem 2.1. *Let M be a null geodesically complete spacetime which obeys the null energy condition,* $\mathrm{Ric}\,(X, X) = R_{ij}X^i X^j \geq 0$, *for all null vectors X. If M admits a null line η then η is contained in a smooth achronal edgeless* totally geodesic *null hypersurface S.*

The simplest illustration of Theorem 2 is Minkowski space: Each null line ℓ in Minkowski space is contained in a unique null hyperplane Π.

We make some comments about the proof, which is based on a maximum principle for C^0 null hypersurfaces, see [8] for details. First, by way of motivation, note that the null plane Π above can be realized as the limit of the future null cone $\partial I^+(x)$ as x goes to past null infinity along the null line ℓ. Π can also be realized as the limit of the past null cone $\partial I^-(x)$ as x goes to future null infinity along the null line ℓ. In fact, one sees that $\Pi = \partial I^+(\ell) = \partial I^-(\ell)$.

Thus, in the setting of Theorem 2, consider the *achronal boundaries* $S_+ = \partial I^+(\eta)$ and $S_- = \partial I^-(\eta)$. By standard causal theoretic results [17], S_+ and S_- are achronal, edgeless, C^0 (but in general not smooth) hypersurfaces in M. Since η is achronal, it follows that S_+ and S_- both contain η. For simplicity, assume S_+ and S_- are connected (otherwise restrict attention to the component of each containing η). The proof then consists of showing that S_+ and S_- agree and form a smooth totally geodesic null hypersurface. We give some indication as to how this works.

By further properties of achronal boundaries (see especially [17, Lemma 3.19]), each point $p \in S_-$ is the past end point of a null geodesic contained in S_- which is future inextendible in M, and hence future complete, i.e., S_- is a C^0 null hypersurface ruled by future complete null geodesics. Similarly, S_+ is a C^0 null hypersurface ruled by past complete null geodesics. Now suppose S_+ and S_- are actually *smooth* null hypersurfaces. Then by standard arguments, based on Raychaudhuri's equation (eq. (2.1)), S_- must have null expansion $\theta_- \geq 0$. Indeed, if $\theta_- < 0$ at some point p then the null generator through p would encounter a null focal point in a finite affine parameter time to the future, forcing the generator to leave S_-, which is impossible. (This is the basis of the proof of the black hole area theorem.) Time-dually, S_+ has null expansion $\theta_+ \leq 0$. Now, let q be a point on both S_+ and S_-. By simple causal considerations one observes that in the vicinity of q, S_+ lies to the future side of S_-. Using the inequalities $\theta_+ \leq 0 \leq \theta_-$, and the fact that the null expansion scalar can be expressed as a second order quasi-linear elliptic operator acting on some function related to the graph of the null hypersurface, one can apply the strong maximum principle to conclude that S_+ and S_- agree near q, c.f., [8, Theorem II.1]. Thus, the nonempty set $S_- \cap S_+$ is both open and closed in S_-, and in S_+. Hence $S_- = S_+$, and this common null hypersurface, call it S, must have vanishing null expansion, $\theta \equiv 0$. Equation (2) then implies that the shear of the generators also vanishes, and hence S is totally geodesic.

The principal difficulty in proving Theorem 2 is in showing that the arguments of the preceding paragraph extend to the C^0 setting. In fact it is possible to show that S_+ and S_- satisfy, $\theta_+ \leq 0 \leq \theta_-$, in a certain weak sense, namely, in the sense of smooth *support* null hypersurfaces. Then, as an application of the weak version of the strong maximum principle obtained in [2] (which includes a regularity assertion), it is again possible to show that S_+ and S_- agree and form a smooth null hypersurface with vanishing null expansion scalar, cf., [8, Theorem III.4]

3. Proof of Theorem 1

A proof of Theorem 1 in the case $\lambda = 0$ was given in [8]. Here we give a proof of Theorem 1, different in a number of respects, which accommodates the case $\lambda > 0$. We also correct a mistake in the proof given in [8]. The proof made use of the erroneous assertion that $M^- = M \cup \mathcal{I}^-$ is causally simple (i.e., that the sets of the form $J^\pm(K, M^-)$ are closed for all compact $K \subset M^-$). As pointed out in [14], without some further assumption on $\mathcal{I}$, this in principle need not hold. In the proof presented here we circumvent the use of causal simplicity.

The first, and main step in proving Theorem 1, is to show that (M, g) has constant curvature. Let η be the assumed null line in M, and let S be the component of $\partial I^+(\eta, M)$ containing η. By Theorem 2, S is a smooth totally geodesic null hypersurface in M. The null line η acquires a past end point p on $\mathcal{I}^-$ and a future end point q on $\mathcal{I}^+$. We examine the structure of S near p. (A similar structure will hold near q.) For this purpose it is convenient to extend $M^- = M \cup \mathcal{I}^-$ slightly beyond its boundary. Thus, without loss of generality, we may assume that M^- is contained in a spacetime (without boundary) P such that $\mathcal{I}^-$ separates P. It follows that $\mathcal{I}^-$ is a globally achronal null hypersurface in P.

Observe that $I^+(\eta, M) = I^+(p, P)$, from which it follows that $\partial I^+(\eta, M) = \partial I^+(p, P) \cap M$ (where $\partial I^+(A, X)$ refers to the boundary in X). Asymptotic simplicity then implies that the generators of $\partial I^+(\eta, M)$ must have past end points on $\mathcal{I}^-$ at p. (A generator γ of $\partial I^+(p, P)$ starting in M either extends to p, or else is past inextendible in P, while remaining in $\partial I^+(p, P)$. In the latter case, γ meets $\mathcal{I}^-$ transversely and enters $I^-(\mathcal{I}^-, P)$, violating the achronality of $\mathcal{I}^-$.) The generators of $S \subset \partial I^+(\eta, M)$ must coincide with those of $\partial I^+(\eta, M)$, and hence have past end points on $\mathcal{I}^-$ at p, as well. Let U be a convex normal neighborhood of p, and let A be the "null cone" in U generated by the future directed null geodesics in U emanating from p. A is a smooth null hypersurface in U, except for the conical singularity at p. We can choose a smooth spacelike hypersurface in U, passing slightly to the future of p, which meets A in a 2-sphere Σ. By invariance of domain, and the fact that S is closed in M, it is easily seen that the subset $\Sigma \cap S$ of $\Sigma \cap M$ is both open and closed in $\Sigma \cap M$. It follows that all the null geodesics forming A correspond precisely to the generators of

S, except, in the asymptotically flat case, for the generator γ_p of $\mathfrak{I}^-$ with past end point p. Note in particular, this implies that $S = \partial I^+(\eta, M)$.

Now, set $N_p = S \cup \gamma_p$ in the asymptotically flat case, and $N_p = S \cup \{p\}$ in the asymptotically de Sitter case. From the above, N_p is made up of all the future inextendible null geodesics in P emanating from p. In the asymptotically de Sitter case, $\mathfrak{I}^-$ is spacelike, so $\partial I^+(p, P)$ meets $\mathfrak{I}^-$ only at p. It follows from statements above that $N_p = \partial I^+(p, P)$. Suppose, in the asymptotically flat case, that $\partial I^+(p, P)$ meets $\mathfrak{I}^-$ at a point x, say. Then there exists a null geodesic $\gamma \subset \partial I^+(p, P)$ with future end point x which either extends in the past to p or is past inextendible in $M \cup \mathfrak{I}^-$. In the latter case, Lemma 4.2 in [14] implies that $M \subset I^+(\gamma)$, from which it follows that $M \subset I^+(p, P)$. But this contradicts the achronality of η. It follows that $\partial I^+(p, P) \cap \mathfrak{I}^- = \gamma_p$, and again we have $N_p = \partial I^+(p, P)$. Thus, N_p is an achronal boundary, and hence is an achronal edgeless C^0 hypersurface in P. Moreover, since the future directed null geodesics in P emanating from p do not cross and, being achronal, do not have points conjugate to p, $N_p \setminus \{p\}$ is the diffeomorphic image under the exponential map $\exp: \mathcal{O} \subset T_p P \to P$ of $(\lambda_p^+ \setminus \{0\}) \cap \mathcal{O}$, where λ_p^+ is the future null cone in $T_p P$ and $\mathcal{O}$ is the maximal open set on which $\exp$ is defined. Thus, apart from the conical singularity at p, N_p is a smooth null hypersurface. We are justified in thinking of N_p as the future null cone in P at p.

The shear tensor σ_{ab} of S in the physical metric g vanishes, and so, since the shear is a conformal invariant, the shear tensor $\tilde{\sigma}_{ab}$ of $N_p \setminus \{p\}$ vanishes in the unphysical metric $\tilde{g}$. Then by the well-known propagation equation for the shear tensor [12, Eq. 4.36], the components $\tilde{C}_{a0b0}$ (with respect to an appropriately chosen frame in which e_0 is aligned along the generators) of the conformal tensor of $\tilde{g}$ vanish on $N_p \setminus \{p\}$. We can now invoke an argument of Friedrich [5] used in essentially the same situation to prove a cosmic no-hair theorem. By use of the regular conformal field equations, specifically the divergencelessness of the rescaled conformal tensor,

$$\nabla_i d^i_{jkl} = 0\,, \qquad d^i_{jkl} = \Omega^{-1} \tilde{C}^i_{jkl}\,,$$

Friedrich shows that the full rescaled conformal tensor d^i_{jkl} vanishes on the future domain of dependence $D^+(N_p, P)$ of N_p. Hence, the conformal tensor with respect to the physical metric vanishes on $D^+(N_p, P) \cap M$. Together with equation (2), this implies that (M, g) has constant curvature on $D^+(N_p, P) \cap M$. In a precisely time-dual fashion (M, g) has constant curvature on $D^-(N_q, Q) \cap M$, where Q is an extension of $M \cup \mathfrak{I}^+$ analogous to P, and N_q is the past null cone in Q at q.

To conclude that M is everywhere of constant curvature we need to show that $M \subset D^+(N_p, P) \cup D^-(N_q, Q)$. Since $S = \partial I^+(\eta, M) = \partial I^-(\eta, M)$, M can

be expressed as the disjoint union,

$$(3.1) \qquad M = I^-(S, M) \cup S \cup I^+(S, M).$$

This decomposition follows from [17, Proposition 3.15], but can be easily shown directly as follows. Let $x \in M \setminus S$. There exists a curve σ in M (not necessarily causal) from x to $y \in S$ that meets S only at y. Either $\sigma \cap I^-(S, M) \neq \emptyset$ or $\sigma \cap I^+(S, M) \neq \emptyset$. If the latter holds and σ is not contained in $I^+(S, M)$, then σ meets $\partial I^+(S, M)$ at some point $z \notin S$. But since $I^+(S, M) = I^+(\partial I^+(\eta, M)) = I^+(\eta, M)$, $z \in \partial I^+(\eta, M) = S$, which is a contradiction. Hence, $\sigma \subset I^+(S, M)$, and so $x \in I^+(S, M)$. Similarly, $\sigma \cap I^-(S, M) \neq \emptyset$ leads to $x \in I^-(S, M)$, which establishes (3.1).

We now show that each term in (3) is a subset of $D^+(N_p, P) \cup D^-(N_q, Q)$. Trivially, $S \subset N_p \subset D^+(N_p, P)$. Consider $I^+(S, M) \subset J^+(N_p, P) \cap M$. We claim that $J^+(N_p, P) \cap M \subset D^+(N_p, P) \cap M$. If not, then $H^+(N_p, P) \cap M \neq \emptyset$. Choose a point $x \in H^+(N_p, P) \cap M$, and let ν be a null generator of $H^+(N_p, P)$ with future end point x. Since N_p is edgeless, ν remains in $H^+(N_p, P)$ as it is extended into the past. By asymptotic simplicity, ν must meet $\mathfrak{I}^-$. In fact, ν must meet $\mathfrak{I}^-$ transversely (even in the asymptotically flat case, since ν starts in M) and then enters $I^-(\mathfrak{I}^-, P)$. But this means that ν has left $J^+(N_p, P)$, which is a contradiction. Hence, $I^+(S, M) \subset D^+(N_p, P)$, and by the time-dual argument, $I^-(S, M) \subset D^-(N_q, Q)$.

Thus, M is globally of constant curvature. By the uniqueness of simply connected Lorentzian space forms, Theorem 1 will follow once we show that M is simply connected and geodesically complete. In the asymptotically flat case, it is shown in [14] that M is simply connected (in fact, is homeomorphic to $\mathbb{R}^4$). Simple connectivity in the anti-de Sitter case may be established as follows. Each generator of S has a past end point at p and a future end point at q. By the structure of S near p and q established above, it follows that $S \cup \{p, q\}$ is homemorphic to the 3-sphere. Perturbing $S \cup \{p, q\}$ slightly near p and q, we obtain a smooth achronal 3-sphere S_0 in M. By an argument similar to one given above, one can show that $H^+(S_0, P) \cap M = H^+(S_0, Q) \cap M = \emptyset$, from which it follows that S_0 is a Cauchy surface for M. (Alternatively, by a suitable modification of arguments in [14], one can show that M is globally hyperbolic. Then, since S_0 is compact and achronal it must be a Cauchy surface.) Thus M has topology $\mathbb{R} \times S_0$, and so is simply connected.

It remains to show that M is geodesically complete. Being asymptotically simple, we know that M is null geodesically complete, which is essential to the proof of full completeness. Let $\bar{M}$ denote Minkowski space in the case $\lambda = 0$, and de Sitter space in the case $\lambda > 0$. By standard results, for each $x \in M$ and $\bar{x} \in \bar{M}$, there is a neighborhood of x isometric to a neighborhood of $\bar{x}$. Since M is simply connected, these local isometries can be pieced together, by an analytic continuation type argument, to produce a local isometry $\phi : M \to \bar{M}$,

see, for example, Theorem 8.17 in [15]. (The assumption in [15, Th. 8.17] that M is complete is not needed to produce the local isometry; it is used only to conclude that ϕ is a covering map.)

Let x be any point in M, and let $\bar{x} = \phi(x)$. Then, from the fact that ϕ is a local isometry and M is null geodesically complete, it follows that any (broken) null geodesic segment in $\bar{M}$ starting at $\bar{x}$ can be lifted uniquely via ϕ to a (broken) null geodesic segment in M starting at x. From this it follows that ϕ is onto: Fix $x \in M$ and let $\bar{x} = \phi(x)$. Let $\bar{\nu}$ be a broken null geodesic in $\bar{M}$ from $\bar{x}$ to any other point $\bar{y}$. Let ν be the lift of $\bar{\nu}$ starting at x, and let y be its final end point; since ν covers $\bar{\nu}$, we must have $\phi(y) = \bar{y}$.

To establish geodesic completeness, it is sufficient to show that any unit speed timelike or spacelike geodesic $\gamma : [0, a) \to M$, $t \to \gamma(t)$, continuously extends to $t = a$. Let $\bar{\gamma} = \phi \circ \gamma$; $\bar{\gamma}$ can be extended to a complete geodesic in $\bar{M}$ which we still refer to as $\bar{\gamma}$. Fixing $u_0 \in [0, a)$ sufficiently close to a, one easily constructs a C^0 homotopy of curves $\{\bar{\sigma}_u\}$, $u_0 \leq u < a$, from $\bar{\gamma}(0)$ to $\bar{\gamma}(a)$, where, for each $u \in [u_0, a)$, $\bar{\sigma}_u$ consists of the segment $\bar{\gamma}|_{[0,u]}$, followed by a suitably chosen two-segment broken null geodesic $\bar{\nu}_u$ from $\bar{\gamma}(u)$ to $\bar{\gamma}(a)$, such that the length of $\bar{\nu}_u$ (in some background Riemannian metric) goes to zero as $u \to a$. The broken null segments $\bar{\nu}_u$ can be constructed from the two families of null geodesics foliating a totally geodesic timelike 2-surface containing $\bar{\gamma}|_{[u_0,a]}$. By the nature of the lifting procedure, the homotopy $\{\bar{\sigma}_u\}$, $u_0 \leq u < a$ can be lifted to a C^0 homotopy $\{\sigma_u\}$, $u_0 \leq u < a$, where, for each u, σ_u consists of the segment $\gamma|_{[0,u]}$, followed by a two-segment broken geodesic ν_u covering $\bar{\nu}_u$. The curves σ_u must have a common final end point q, say. Since the length of ν_u goes to zero (in the lifted Riemannian metric) as $u \to a$, it follows that $\gamma(u) \to q$ as $u \to a$, i.e., γ is extendible to q. This completes the proof of Theorem 1.

4. Concluding Remarks

It is possible to formulate a version of Theorem 1 applicable to anti-de Sitter space. However, since in this case $\mathcal{I}$ is timelike, the characteristic initial value problem encountered in the proof of Theorem 1 would become an initial-boundary value problem, and it would be necessary to impose boundary data (namely the vanishing of the conformal tensor) on $\mathcal{I}$. In this case one might expect to establish the uniqueness of anti-de Sitter space without the assumption of a null line.

We believe that one should be able to generalize Theorem 1 in certain directions. For example, in the asymptotically flat case, we expect that the vacuum assumption could be weakened to allow for the presence of matter. For instance, if we assume spacetime M satisfies the null energy condition, and is vacuum in a neighborhood of $\mathcal{I}$, then the proof of Theorem 1 implies the existence of open sets near scri which are flat. If the vacuum region is analytic then it follows that

a neighborhood of scri is flat, and so M should have vanishing mass. By a suitable version of the positive mass theorem, M should be isometric to Minkowski space. Thus, we conjecture that, in the asymptotically flat case, the vacuum assumption can be replaced by the null energy condition and a requirement that the spacetime Ricci tensor falls off at an appropriate rate on approach to $\mathcal{I}$. It may also be possible, in the asymptotically flat case, to prove a version of Theorem 1 for *weakly* asymptotically simple spacetimes, which allows for the occurence of black holes. By imposing suitable conditions on the domain of outer communications $I^-(\mathcal{I}^+) \cap I^+(\mathcal{I}^-)$ (e.g., that the DOC be globally hyperbolic) one might expect to be able to show that the DOC is flat. To accomplish this, one may be able to exploit the fact that the proof of the null splitting theorem does not actually require the full null completeness of spacetime.

We mention in closing that the null splitting theorem has recently been used [1] to obtain restrictions on the topology of $\mathcal{I}^\pm$ in asymptotically simple and de Sitter spacetimes obeying the null energy condition.

Acknowledgement

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